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The Gauss's, Theorema Egregium

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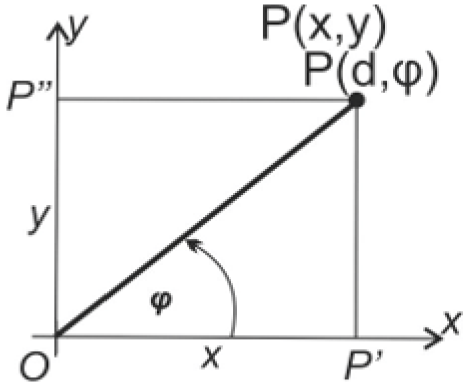
Abstract:
To understand Gauss's theory about the non-Euclidean geometry we have to reestablish some definitions of the coordinate system, and introduce the so-called Gaussian coordinates. We show here that the two points distances as a postulate can establish a metric geometry. If we can show the validity of this postulate on any surface than it has his geometry, and not necessarily Euclidean. Gauss showed in The Theorema Egregium that a surface might have such attributes. The different geometries of the regular surfaces written here are Euclidean, spherical, and hyperbolic. This theorem presented in 1827. (Based on the lectures of K. Lanczos: Department of Physical Sciences and Applied Mathematics, North Carolina State University, Raleigh, 1968.) The importance of this lecture is to make clear and understandable by using Gausss's theorem how and why the physicians must use non-Euclidean geometry.

ANTECEDENTS

A Postulate of The Coordinate System Establish the Metric Geometry

The Cartesian coordinate system applicable for the full Euclidean geometry and every point is metric: the distance between any two points determined by the algebraic way. In other words, the Cartesian coordinate system and all of its correct conversion are metric spaces and exist such a portion, which is fully met with the Euclidean geometry [3]. Gauss showed that the full geometry could be constructed by only one postulate.



Interpretation

The distance between any two points AB is: $s^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$.
Gauss showed that the Euclidean geometry can be deduced from this postulate.

We study here only the structure and validity of the coordinate system based on this postulate.

Axes:
Straight lines, which intersect each other at the origin and perpendicular pairwise. Such a line, for example, the number line with the ordered set of the real numbers. Two perpendicular axes form

a plane, which the privileged point is the origin *O*, then we call them respectively, *x-axis* or abscissa and *y-axis* or ordinate.

Coordinates:
Distances measured from the axes, in geometric terms a perpendicular projection of the point *P* to the axes. The distance is the length of this section.

The point now a pair: $P(x,y) \quad \left| x = \overline{PP''} = \overline{OP'}; y = \overline{PP'} = \overline{OP''} \right.$

The relationship between the pair and the point is mutual: we can define *x and y* from the point *P*, or vice-versa get the *P* point from *x and y*.

Consequences:

1. We may replace all geometric constructions with algebraic operations.
2. We may replace any algebraic operation for an *(x,y)* to geometric construction.

***Angle**:*
Gives the direction of the straight line, that goes through O and P points, and we usually measure it from the *x*-axis. Geometrically the inclination of two lines lays both on origin, i.e. the angle between them. Algebraically the ratio of the *P* coordinates: $tg\alpha = \frac{y}{x} \rightarrow \alpha = arctg\frac{y}{x}$ also called *tangent*, whereas *POP'* is a right triangle (the angle at the origin).

***Straight-Line**:*
We can get the straight-line equation if we use $\frac{y}{x} = tg\alpha = m$ for a point *P* lay on a line: $y = mx$, which means that any solutions of this equation – the *(x,y)* pairs - are on the line. If the line does not cross the origin, then the equation altered to: $y = mx + y_0$, where y_0 the cross point with the *y*-axis, or algebraically the equation's solution for *x=0*.

We show here that the algebraic expression $Ax + By + C = 0$ - a linear equation with two unknown quantities - all possible solution's lay on a straight line and also describe all its points.

The $y = mx + y_0$ straight-line lays on two points: $(0, y_0)$ and $(x_0, 0)$ which intersects the axes. So, from the two equations $m = tg\frac{y_0}{x_0}$ and $y = \frac{y_0}{x_0}x + y_0$, by making common denominator $y = \frac{y_0 x + y_0 x_0}{x_0}$ and reduce to zero, we get: $y_0 x - x_0 y + x_0 y_0 = 0$. Now let us use these notions: $A = y_0; B = -x_0; C = x_0 y_0$ (see the meaning of the negative value on the figure: if $y_0 > 0$, then $x_0 < 0$ and vice versa). Thus, we showed that the $Ax + By + C = 0$ expression is the coordinate-geometry form of the straight line. Any points on the line are solutions of the equation and only those.

Circle:
The definition of the circle immediately gives its equation: a geometric location of all points, which lay in the same distance from a common point. So, using the distance postulate: $r^2 = (x - x_0)^2 + (y - y_0)^2$, where *r* the radius and (x_0, y_0) is the origin.

Arc:
A piece of the circle line. The angle of the full circle is *2π*, and the circumference of a circle is *2rπ*, then proportionally the AB arc has an angle φ_{AB} and so $\varphi = \frac{AB}{r}$, and its length $AB = \varphi r$.

Now we have shown that the basic elements of the Euclidean geometry fully revealed in a Cartesian coordinate system.

Infinitesimal Distance:
for the sake of generalization of the space concept, we satisfy to study the immediate surrounding of a point, so we interpret the distance between (x, y) and $(x+\Delta x, y+\Delta y)$ where the Δ can be any small size, i.e., infinitesimal. Then using our only postulate, the distance: $ds^2 = dx^2 + dy^2$, because $x + dx - x = dx$ and $y + dy - y = dy$.

The derivate of the function $y = f(x)$ is: $f'(x) = \frac{dy}{dx} \rightarrow dy = f'(x)dx \rightarrow$ then $dy^2 = f'^2(x)dx^2$ and from these $ds^2 = dx^2 + f'^2(x)dx^2$. We have the distance by integration between the two points: $s = \int_A^B \sqrt{1 + f'^2(x)}dx$ if *s* minimal. Leaving the details of the reduction – requires variation computing - we arrive at the line equation: $Ax + By + C = 0$, which means, the smallest distance between two points is a straight-line segment. This refers to our geometric attitude. So, we proved that the postulate valid in the infinitesimal environment also.

The Intersection of Two Lines:
Very interesting task to find the intersection of two lines.

Let us have these two lines:

$$e_1 : a_1x + b_1y + c_1 = 0 \text{ and } e_2 : a_2x + b_2y + c_2 = 0 .$$

The *P* interception point is the common solution of the two equations, the *(x,y)* value that satisfies both equations. Leaving the reduction out we get: $x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}$ and $y = \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}$.

Consequently, the two lines always intersect each other if the determinant non-zero: $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$.

If it where rather zero: $a_1b_2 - a_2b_1 = 0$, then $\frac{a_1}{a_2} = \frac{b_1}{b_2}$, which means the tangents are equal because $m_1 = \frac{a_1}{a_2}$ and $m_2 = \frac{b_1}{b_2}$. Thus, the two lines are parallel i.e., no common points.

Curve Line Coordinates
We arrived at beautiful results, then we continue with our imaginations, and we assume that the lines replaced by arbitrary curves like this: *x = f (t)* and *y = g (t)*, where *f (t)* and *g (t)* is continuous function of variable *t*, and ought to be differentiable for infinitesimal use.

There have been examined already many curves geometrically – usually, each is special case - but we would like to arrive at a general solution by algebraic way. It is possible, as we did not attach any more condition, only the continuity and differentiability. Then in this way, we can determine the 'direction' of a curve and introduce the concept of 'curvature'.

The direction of a curve at any point is the gradient of the tangent line: $ds^2 = dx^2 + dy^2 \to 0$ drawn to that point. The direction changed from point to point: this is what we call: curvature. Now, draw a circle through three points of the curve – the best fitting circle – then the distance from its origin will be proportional to the curvature at that point. If we determine these origins for all points, we get another curve with ordinates: $\xi = \varphi(t)$, $\eta = \theta(t)$. This we call *evolute* of the original curve.

Gaussian Coordinates

As we have seen the geometric problems translated to algebraic ones by using, either orthogonal (x, y) or polar (r, φ) coordinates. Then their conversion: $x = r\cos\varphi ; y = r\sin\varphi$ [3]. We also have seen the coordinate-lines dividing a plane into small quadrants.

Let's consider these generally, according to Gauss, and introduce the following general relations: $x = x(u,v) ; y = y(u,v)$ which again have to be continuous and differentiable in the studying environment, and have also a non-zero determinant: $\begin{vmatrix} \frac{\delta x}{\delta u} & \frac{\delta x}{\delta v} \\ \frac{\delta y}{\delta u} & \frac{\delta y}{\delta v} \end{vmatrix} \neq 0$. Note that this not met at *r=o*, in the origin's environment.

The so introduced *(u,v)* pairs uniquely define the points of a surface. These referred to as Gaussian coordinates. The coordinate-lines drew according to *(u,v)* also divides the plane into small quadrants.

Now we have to show that the postulate also valid with the Gauss-coordinates. According to the determinant above: $dx = \frac{\delta x}{\delta u}du + \frac{\delta x}{\delta v}dv$; $dy = \frac{\delta y}{\delta u}du + \frac{\delta y}{\delta v}dv$ and by this the distance i.e., the postulate is

$$ds^2 = dx^2 + dy^2 = \left[\left(\frac{\delta x}{\delta u}\right)^2 + \left(\frac{\delta y}{\delta u}\right)^2\right]du^2 + \left[\left(\frac{\delta x}{\delta v}\right)^2 + \left(\frac{\delta y}{\delta v}\right)^2\right]dv^2 + 2\left(\frac{\delta x}{\delta u}\frac{\delta x}{\delta v} + \frac{\delta y}{\delta u}\frac{\delta y}{\delta v}\right)dudv$$

In the case of polar coordinates, the expression with the necessary reductions is: $ds^2 = dr^2 + r^2 d\varphi^2$. Now the arc, which is the shortest way between any two points *A*, *B* gives: $(a\cos\varphi)r + (b\sin\varphi)r + c = 0$. Therefore, this is the postulate!

Now we proved that in both, the Cartesian and the Gaussian coordinates - straight line, curve line or orthogonal and non-orthogonal – the distance postulate valid and describes the full geometry.

Now we may expand the here discussed two-dimensional space to any space if the postulate remains unchangeably valid in the resulting space. We call these spaces *metric-space* according to our modern conceptions, regardless of the number of dimensions.

In many cases, we will be satisfied if the conditions apply only in the immediate surroundings of a point in a space. Conversely, if any points in a space have such an environment by which a coordinate system interpreted, then the space is a *Euclidean topological space*.

Gauss's Non-Euclidean Idea

Gauss came to an interesting result when he had to perform measurements in a hilly area. Provided two sets of curves intersecting each other mutually, like the coordinate-lines. These are the already known *(u,v)* pairs. Now, if we place them into a three-dimensional orthogonal coordinate system then it is expressed this way $x = x(u,v)$; $y = y(u,v)$; $z = z(u,v)$, and the arc:

$$ds^2 = dx^2 + dy^2 + dz^2 .$$

Express the former with the latter:

$$dx = \frac{\delta x}{\delta u} du + \frac{\delta x}{\delta v} dv \text{; } dy = \frac{\delta y}{\delta u} du + \frac{\delta y}{\delta v} dv \text{; } dz = \frac{\delta z}{\delta u} du + \frac{\delta z}{\delta v} dv$$

and replace to the arc expression, then we get $ds^2 = Edu^2 + 2Fdudv + Gdv^2$, where

$$E = \left(\frac{\delta x}{\delta u}\right)^2 + \left(\frac{\delta y}{\delta u}\right)^2 + \left(\frac{\delta z}{\delta u}\right)^2 ;$$
$$F = \frac{\delta x}{\delta u}\frac{\delta x}{\delta v} + \frac{\delta y}{\delta u}\frac{\delta y}{\delta v} + \frac{\delta z}{\delta u}\frac{\delta z}{\delta v} ;$$
$$G = \left(\frac{\delta x}{\delta v}\right)^2 + \left(\frac{\delta y}{\delta v}\right)^2 + \left(\frac{\delta z}{\delta v}\right)^2 .$$

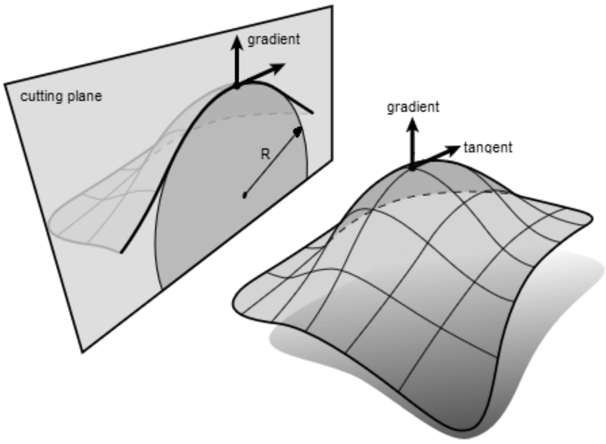
We come to an interesting result: the distance on the surface and in the space may have the same. Rather these points are on a curve on the surface and on a straight-line segment in the space. However, it can be true infinitesimally, as the two points are arbitrarily close to each other. This ds^2 is a limit and a common value in the space and on the surface. We got a different geometry, the *internal geometry of the surface*, where these shortest lines are straight. Moreover, it shall remain valid as long as we stay on the surface.

Therefore, Gauss showed that the internal geometry of a curve surface is uncontradictory and does not have to be the subject of the Euclidean postulates. If this surface is an ellipsoid, for example, then easy to understand that a triangle will incongruent for the move, either the sides or the angles will change. The consequences are that the space changes from point to point.

THE THEOREM EGREGIUM

The Curvature

In 1827, Gauss published the *Disquisitiones generales circa superficies of the curves*, - General studies of the Curved Surfaces [4] - and inside with a theorem that he signed as remarkable: *Theorem Egregium*.



This writing defined the curvature as follows:
Let given a surface, and construct the gradient of the tangent plane for a point *P*. Now we use a plane along the gradient, which will cut a plane-curve from the surface. If we move around this plane along the gradient as an axis, then in each step the cut of the plane-curve will be different. We get dissimilar radiuses and curvatures. However, each curvature will have a maximum then a minimum value, in the extreme positions of the rotating plane: radius R_1 *and* R_2. Let's call the reciprocals as *curvatures* and the extremes as *main curvatures*: $k_1 = \frac{1}{R_1}$ and $k_2 = \frac{1}{R_2}$. Of course, this may not be an internal property of the surface, since *the gradient is outside from the surface*. Therefore, the curvature is available only from a space that contains the surface.

Consider the product of the two *main curvature* $k = k_1 k_2 = \frac{1}{R_1 R_2}$.

Gauss came to the surprising conclusion, - which is not deducted here – that the value of *k* can get from *(E), (F), and (G).*

Then, this is notwithstanding an *internal property of the surface*, regardless of whether we defined externally. The value of *k* independent from the *(u,v)* coordinates since we constructed it with clear geometry. *So, k, the curvature is invariant in any Gaussian coordinate-system!*

The Property of k

Now examine the different values of *k.*

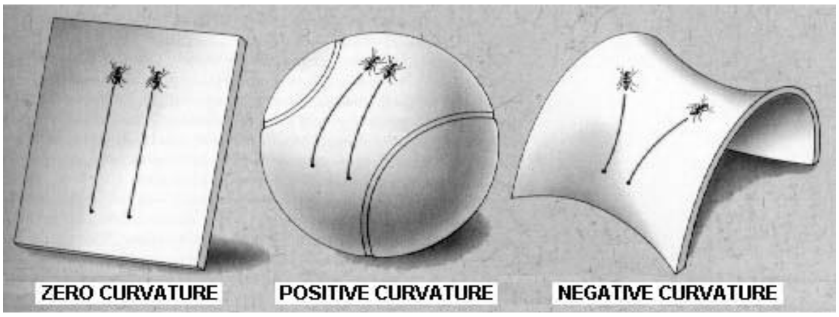
In general cases, the value of *k* constantly changes according to the surface, but let's examine the special cases:

- If *k=zero*, then the surface is a plane, become Euclidean;
- If *k=constant*, then the surface is even and so the forms on it may freely move without changes.

- The constant value either positive or negative:
- If the radiuses are on the same side – positive *k* - of the tangent - than the surface convex;
- If they are on different side, – negative k - then the surface saddle-shaped ie. in all directions move away from point *P*.

As we already mentioned, we are talking about, "even" surface, i.e., the curvature constant. We also did the necessary calculations. Then let us, have a unit size curvature, and then we come to the following terms. If

k = 1, then the geometry spherical: $ds^2 = du^2 + dv^2 \sin^2 u$
k = 0, then the geometry Euclidean: $ds^2 = du^2 + dv^2 u^2$
k =-1, then the geometry hyperbolic: $ds^2 = du^2 + dv^2 sh^2 u$



Different *k* values

Notice the simple differences between each distance and yet they open a quite different world. The *Euclidean Geometry* has the multiplier factor: u^2 , the *Spherical Geometry* has the: $\sin^2 u$, and the *Hyperbolic Geometry* has the: $sh^2 u$.

This is the beauty of mathematics.

Other Results

Gauss in this presentation came to the definition of the non-Euclidean geometries, having evidence about their existence and uncontradictory.

Gauss's investigations covered this field are little known, but we know his other result which leads to the sum of the angles in a triangle. This definition uses the calculation of the area of a triangle, which is a relationship between the area and the curvature. He came to the following: $\alpha + \beta + \gamma - \pi = \int k d\delta$, where the $d\delta$ is an infinitesimal surface-unit and the integration gives the area of the entire triangle. From the expression using the three constant *k* values we get these:

k=1 $\alpha + \beta + \gamma - \pi = \Delta$
k=0 $\alpha + \beta + \gamma = \pi$
k=-1 $\pi - (\alpha + \beta + \gamma) = \Delta$

This means that the area of a triangle is proportional to the sum of its internal angles. So, this emerges that in spherical case *>180°*, while in the hyperbolic case, *<180°*, and we get back the Euclidean case if *=180°.*

A further result is that this area calculation can be applied also in general case - in an infinitesimal sense - even when the *k* changes point by point.

CONCLUSION

Gauss's work is not the composition of the non-Euclidean geometry; however, these results undoubtedly deserve the *remarkable or prominent theory* name. Consequently, Gauss's name suitable beside the names of Bólyai and Lobachevski.

In addition, the consequences of this theorem led us to Riemann Geometry.

For the physicians means that if any physical space or motion describable with Gaussian coordinates may calculate according to non-Euclidean geometry and vice-versa.

This has special significance for me because I am not a fan of scientific racing or star making. I rather much believe in the more effective work, make it by one or plenty, anyone. This statement is extremely important these days when science has also promoted collaboration. Please remember to all participants and not let just the leaders win the glory and have the recognition.
I know it's not easy, though only this is worthy.

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